A HELLY CONVERGENCE THEOREM FOR STIELTJES INTEGRALS

BY

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1. *Introduction*

Let E be a simply discontinuous function which is right-continuous on an interval $[a, b)$; moreover, E is assumed to be continuous at the end-point $b \leqslant \infty$ (we allow $a = -\infty$, and the range of E lies in a locally convex vector space). Suppose that g belongs to the family (BV) of all complex-valued functions of bounded variation on $[a, b]$. Under these circumstances, it will be shown that the Stieltjes sums

$$\sum_{k=1}^{m} g(x_k{}^*) \cdot (E(\bar{x}_k) - E(\bar{x}_{k-1}))$$

(with $x_k{}^* = \bar{x}_k$) converge to a limit, the so-called *"right Cauchy integral"*, which we denote

$$\int g(\oplus[\lambda]) \cdot dE(\lambda); \tag{1}$$

the limit is to be understood in the sense of refinements of subdivisions of $[a, b]$. If g is left-continuous, then

$$\int g(\lambda) \cdot dE(\lambda) = \int g(\oplus[\lambda]) \cdot dE(\lambda), \tag{2}$$

where the left-hand side is the ordinary Stieltjes refinement integral – whose definition can be obtained as follows: replace by $\bar{x}_{k-1} \leqslant x_k{}^* \leqslant \bar{x}_k$ the reuirement $x_k{}^* = \bar{x}_k$ in the definition of the integral (1). If $E(a) = \bigcirc$ and $g \in (BV)$, then

$$\int E(i[\lambda]) \cdot dg(\lambda) = g(b) \cdot E(b) - g(a) \cdot E(a) - \int g(\oplus[\lambda]) \cdot dE(\lambda); \tag{3}$$

the left-hand integral is what HILDEBRANDT [9, p. 273] calls a "modified σ-Stieltjes integral" – replace by $\bar{x}_{k-1} < x_k{}^* < \bar{x}_k$ the requirement $x_k{}^* = \bar{x}_k$ in the definition of the integral (1). Relation (3) is an immediate consequence of our main theorem (2.10).

Theorem A. Let $(g_s : s \in \sigma)$ be a generalized sequence in (BV) such that

(i) $\quad \infty \neq \sup_{s \in \sigma} \operatorname{var} g_s$

(here var = total variation). Suppose that $h \in (BV)$ and that

(ii) $\qquad h(\lambda) = \lim_{s \in \sigma} g_s(\lambda) \qquad$ (when $a < \lambda \leqslant b$).

If $E(a) = \bigcirc$, then

(iii) $\qquad \int h(\oplus[\lambda]) \cdot dE(\lambda) = \lim_{s \in \sigma} \int g_s(\oplus[\lambda]) \cdot dE(\lambda),$

and

(iv) $\qquad \int E(i[\lambda]) \cdot dh(\lambda) = \lim_{s \in \sigma} \int E(i[\lambda]) \cdot dg_s(\lambda).$

In case h and all the g_s are left-continuous, then (iii)–(iv) *are valid for ordinary Stieltjes refinement integrals.*

The above theorem can be derived from our main results (2.10); see 4.4. In a way, property (iv) is stronger than the conclusion of the original HELLY theorem ([7] and [8, p. 70]).

For the purpose of subsequent applications to spectral theory, we suppose that the range of E is an equicontinuous family of linear operators; spectral resolutions are such functions, and the following example shows that E need not be of bounded variation in any metric sense of the word: let E be the extension to l_p of the spectral resolution of the unitary shift operator on l_2 (see 5.2 in [12]); here $1 < p < \infty$ and the value of the integral

$$\int e^{2\pi i \lambda} \cdot dE(\lambda)$$

is the unitary shift operator on l_p; the range of E is an equicontinuous subset of the space $\mathscr{A}$ of all bounded endomorphisms of l_p. In this example, $\mathscr{A}$ must be endowed with the strong operator topology in order to ensure that E satisfy our hypotheses, and $\mathscr{A}$ then fails to be complete.

Our hypotheses include the particular case where E has its values in an arbitrary *complete* locally convex vector space. The main results are gathered in 2.10. The presentation is self-contained: included are details and calculations that are familiar in Riemann–Stieltjes integration theory (these have to be re-examined, since the range of E need not lie in a complete space). Concerning relation (3), a counterexample constructed by DUSHNIK [4] shows that (3) is false when $(i[\lambda])$ replaces $(\oplus[\lambda])$; MAC NERNEY [15] has recently announced an integration-by-parts formula differing from (3) by the presence of a Young integral on the right-hand side. If g is left-continuous, then (3) is the integration-by-parts formula for refinement integrals.

I wish to acknowledge my gratitude to T. H. Hildebrandt who sent me his own elegant proof (for complex-valued E) in addition to detailed information. To a 1959 Pacific J. of Math. referee of the article [11] I owe the idea of integrating with respect to continuous spectral resolutions that need not be of bounded variation — the present article does not require that the integrator be continuous. In this connection, the following

fact may be mentionned without proof: if either the integrand or the integrator is continuous, then the integral converges in the uniform operator topology.

2. *Preliminaries, main theorem*

This paper is mainly devoted to the proof of a theorem formulated in 2.10; the intervening paragraphs contain definitions and notational conventions. Fixed throughout is an interval $[a, b]=\{\lambda : a\leqslant\lambda\leqslant b\}$ with $-\infty\leqslant a<b\leqslant\infty$.

2.1. A "*subdivision*" is a strictly-increasing sequence $u=(u_k : 0\leqslant k\leqslant m)$ such that

$$a=u_0<u_1<\ldots<u_{k-1}<u_k<\ldots<u_m=b;$$

let $M(u)$ denote the family of all sequences $(x_k{}^* : k=1,\ldots,m)$ such that $u_{k-1}\leqslant x_k{}^*\leqslant u_k$, and set

$$||u||=\max\{u_k-u_{k-1} : k=1,\ldots,m\}. \tag{4}$$

If $w=(w_i : 0\leqslant i\leqslant n)$ is a subdivision, we write $u\geqq w$ if (and only if) each closed interval $[u_{k-1}, u_k]$ is contained in some interval $[w_{i-1}, w_i]$. The family U of all subdivisions is directed by the relation $\geqq$.

2.2. A "*marked partition*" is a pair (u, x^*) such that $u\in U$ and $x^*\in M(u)$; let (π) denote the family of all marked partitions. Thus, $x\in(\pi)$ if x is a pair $(\bar{x}, x^*)$ of sequences $\bar{x}=(\bar{x}_k : 0\leqslant k\leqslant m)\in U$ and $x^*=(x_k{}^* : k=1,\ldots,m)$ such that

$$a=\bar{x}_0\leqslant x_1{}^*\leqslant\bar{x}_1\leqslant\ldots\leqslant\bar{x}_{k-1}\leqslant x_k{}^*\leqslant\bar{x}_k\leqslant\ldots\leqslant\bar{x}_m=b;$$

An ordering relation $\geqq$ is defined on (π) as follows:

$$(\bar{x}, x^*)\geqq(\bar{y}, y^*) \text{ if (and only if) } \bar{x}\geqq\bar{y}; \tag{5}$$

The set $\{\bar{x}_k : 0\leqslant k\leqslant m\}$ will be called the "*range*" of x; it is easily verified that $x\geqq y$ if (and only if) the range of x is contained in the range of y.

2.3. Let Φ and Ψ be vector-valued functions, and let one of them be complex-valued (so the product $\Phi(\lambda)\cdot\Psi(\lambda)$ has a meaning when $\lambda\in[a, b]$). The following abbreviation will be used whenever x is a marked partition with range $\{\bar{x}_k : 0\leqslant k\leqslant m\}$:

$$\Sigma\Phi\cdot(\Psi\Delta x)=\sum_{k=1}^{m}\Phi(x_k{}^*)\cdot(\Psi(\bar{x}_k)-\Psi(\bar{x}_{k-1})). \tag{6}$$

2.4. Two subsets of (π) will now be defined. Let $(\pi\oplus)$ be the family of all $(\bar{x}, x^*)\in(\pi)$ such that $x_k{}^*=\bar{x}_k$ for all $k\geqslant1$ in the domain of $\bar{x}$. Let (πi) denote the family of all $(\bar{x}, x^*)\in(\pi)$ such that $\bar{x}_{k-1}<x_k{}^*<\bar{x}_k$ for all $k\geqslant1$ in the domain of $\bar{x}$. All three sets (π), $(\pi\oplus)$, (πi) are henceforth associated the ordering relation $\geqq$; they are directed sets. Consequently, the mapping

$$(\pi) \ni x \to \Sigma\Phi \cdot (\Psi \Delta x)$$

is a "*generalized sequence*" (terminology of [3, p. 26]); its limit is the ordinary Stieltjes refinement integral (also called the σ-integral [8, 9]); using the notation of [10, p. 68] we write

$$\int \Phi(\lambda) \cdot d\Psi(\lambda) = \lim \{\Sigma\Phi \cdot (\Psi \Delta x) : x \in (\pi), \geqq\}. \tag{7}$$

On the other hand, the so-called "*right Cauchy integral*" is defined as follows:

$$\int \Phi(\oplus[\lambda]) \cdot d\Psi(\lambda) = \lim \{\Sigma\Phi \cdot (\Psi \Delta x) : x \in (\pi\oplus), \geqq\}; \tag{8}$$

it has been studied by Bzoch [1], Schärf [16] and Mac Nerney [14]. We also write

$$\int \Phi(i[\lambda]) \cdot d\Psi(\lambda) = \lim \{\Sigma\Phi \cdot (\Psi \Delta x) : x \in (\pi i), \geqq\}; \tag{9}$$

this integral is also called the "*modified σ-Stieltjes integral*" [8], the "*Dushnik integral*" [6], or the "*interior*" integral [14, p. 317]. The main properties of all three integrals are surveyed in [8, 9].

2.5. Let (BV) be the family of all complex-valued functions of bounded variation on $[a, b]$, and let G be the family of all $g \in (BV)$ such that

$$g(\lambda - 0) = g(\lambda) \qquad (\text{whenever } a < \lambda \leqslant b). \tag{G}$$

The ordinary integral

$$\int E(\lambda) \cdot dg(\lambda) \tag{10}$$

need not exist when $g \notin G$, and will then be replaced by the more general integral

$$\int E(i[\lambda]) \cdot dg(\lambda); \tag{11}$$

n order to handle both situations simultaneously, we shall write

$$\pi(g, i) = \begin{cases} (\pi i) & \text{if } g \notin G \\ (\pi) & \text{if } g \in G, \end{cases} \tag{12}$$

and set

$$\oint E \cdot dg = \lim \{\Sigma E \cdot (g \Delta x) : x \in \pi(g, i), \geqq\}. \tag{13}$$

This amounts to defining (13) to be the integral (11) except when $g \in G$, in which case (13) is the ordinary integral (10).

2.6. Let $\mathscr{A}$ be a locally convex Hausdorff space whose topology is defined by a family $\{|\cdot|_r : r \in r_0\}$ of semi-norms. The space $\mathscr{A}$ is said to be "complete" if, for an arbitrary generalized sequence $(A_s : s \in \sigma, \succ)$ in $\mathscr{A}$, the relation

(14) $$0 = \lim_{y,z} |A_y - A_z|_r \qquad \text{(for all } r \text{ in } r_0)$$

implies the existence of some $B \in \mathscr{A}$ with

(15) $$B = \lim \{A_s : s \in \sigma, \succ\}.$$

The cumbersome notation in (15) will often hereafter be replaced by the notation

$$\lim_s A_s;$$

B is the unique element of $\mathscr{A}$ such that any $\varepsilon > 0$ and any $r \in r_0$ give rise to an element s_r of σ such that $|B - A_s|_r \leqslant \varepsilon$ whenever $s \in \sigma$ and $s \succ s_r$.

2.7. If E belongs to the family $\mathscr{E}_0[\mathscr{A}]$ of all functions defined on $[a, b]$ with values in $\mathscr{A}$, then $E(\lambda) \in \mathscr{A}$ for $\lambda \in [a, b]$; we write

$$E(\lambda \pm 0) = \lim \{E(\beta) : \beta \to \lambda \pm 0\}.$$

Let $\mathscr{E}(\mathscr{A})$ be the family of all $E \in \mathscr{E}_0[\mathscr{A}]$ that satisfy the following three conditions:

(v_1) the limit $E(\lambda - 0)$ exists whenever $a < \lambda \leqslant b$,

(v_2) $E(b-0) = E(b)$,

(v_3) $E(\lambda) = E(\lambda + 0)$ whenever $a \leqslant \lambda < b$.

2.7.1. Let M_{00} be a family of semi-norms determining the topology of a locally convex Hausdorff space $\mathfrak{M}$, and let $\mathscr{N}$ be a locally convex Hausdorff space whose topology is determined by a family $\{||\cdot||_j : j \in j_0\}$ of semi-norms. If $\mathscr{S}$ is a set of linear mappings of $\mathfrak{M}$ into $\mathscr{N}$, we say that $\mathscr{S}$ is "equicontinuous" if (and only if) any $j \in j_0$ gives rise to a number $k_j > 0$ and a finite family $M_j \subset M_{00}$ such that

$$||A(m)||_j \leqslant k_j \cdot \max \{p(m) : p \in M_j\} \qquad (\text{all } A \in \mathscr{S},\ m \in \mathfrak{M}).$$

Let $\mathscr{L}(\mathfrak{M}, \mathscr{N})$ denote the space of all continuous linear mappings of $\mathfrak{M}$ into $\mathscr{N}$: note that $A \in \mathscr{L}(\mathfrak{M}, \mathscr{N})$ if (and only if) the operator A is linear and the one-element family $\{A\}$ is equicontinuous.

2.7.2. When $\mathscr{A} = \mathscr{L}(\mathfrak{M}, \mathscr{N})$, then $\mathscr{A}$ shall henceforth be endowed with the strong operator topology. When $r = (m, j) \in \mathfrak{M} \times j_0$ we set

(16) $$|A|_r = ||A(m)||_j \qquad (\text{whenever } A \in \mathscr{A});$$

the topology of $\mathscr{A}$ is defined by the family $\{|\cdot|_r : r \in r_0\}$ of semi-norms, where $r_0 = \mathfrak{M} \times j_0$. Denote by $\mathscr{F}(\mathscr{A})$ the family of all functions $E \in \mathscr{E}(\mathscr{A})$ such that the range of E is equicontinuous. In other words, $E \in \mathscr{E}(\mathscr{A})$

if E satisfies the three conditions (v_1)–(v_3) and if any $j \in j_0$ gives rise to a number $j[E]>0$ and a finite family $M_j \subset M_{00}$ such that

$$(v_4) \qquad \sup_{a \leqslant \lambda \leqslant b} \|E(\lambda)(m)\|_j \leqslant j[E] \cdot \max\{p(m) : p \in M_j\} \qquad (\text{all } j \in j_0,\ m \in \mathfrak{M}).$$

In the particular case where $\mathfrak{M}$ is a Banach space with norm $\|\cdot\|_p$, the "*operator semi-norm*" is defined as follows:

$$\|E(\lambda)\|_{j,p} = \sup\{\|E(\lambda)(m)\|_j : m \in \mathfrak{M} \text{ and } \|m\|_p \leqslant 1\},$$

and condition (v_4) becomes:

$$(17) \qquad \infty \neq \sup_{a \leqslant \lambda \leqslant b} \|E(\lambda)\|_{j,p} \qquad (\text{all } j \in j_0);$$

but the uniform boundedness theorem now shows (17) to be a consequence of the fact that the range of E is a bounded subset of $\mathscr{L}(\mathfrak{M}, \mathscr{N})$. In other words: $\mathscr{F}(\mathscr{A}) = \mathscr{E}(\mathscr{A})$ when $\mathscr{A} = \mathscr{L}(\mathfrak{M}, \mathscr{N})$ and $\mathfrak{M}$ is a Banach space.

2.8. For example, $\mathscr{A}$ could be the complex plane C. If $A \in \mathscr{A}$ and $t \in C$, then $t \cdot A$ denotes the scalar product; thus $t \cdot A = \bigcirc$ (=the zero element of $\mathscr{A}$) whenever $t=0$. Suppose that $f \in \mathscr{E}_0(C)$ and $A \in \mathscr{A}$: the relation

$$(18) \qquad (Af)(\lambda) = f(\lambda) \cdot A \qquad (\text{for all } \lambda \in [a, b])$$

defines a function $Af \in \mathscr{E}_0(\mathscr{A})$; in fact, it is easily seen that $Af \in \mathscr{E}(\mathscr{A})$ whenever $f \in \mathscr{E}(C)$.

2.9. If $E \in \mathscr{E}_0(\mathscr{A})$, then (13) defines an element of $\mathscr{A}$, and the relation $B = \oint E \cdot dg$ holds if (and only if) any $\varepsilon > 0$ and any $r \in r_0$ give rise to a subdivision u such that

$$|B - \Sigma E \cdot (g \Delta x)|_r \leqslant \varepsilon$$

whenever $x \in \pi(g, i)$ and $\bar{x} \geqq u$. Two immediate consequences: if E_1 and E_2 belong to $\mathscr{E}(\mathscr{A})$, then

$$(19) \qquad \oint (E_1 + E_2) \cdot dg = \oint E_1 \cdot dg + \oint E_2 \cdot dg, \quad \text{and}$$

$$(20) \qquad \oint Af \cdot dg = (\oint f \cdot dg) \cdot A \qquad (\text{all } A \in \mathscr{A} \text{ and } f \in \mathscr{E}(C)).$$

The existence of these integrals is ensured by the following

2.10 Theorem B. *Let $\mathfrak{M}$ and $\mathscr{N}$ be locally convex Hausdorff vector spoces, and suppose that $\mathscr{N}$ is complete; $\mathscr{A}$ is the result of endowing $\mathscr{L}(\mathfrak{M}, \mathscr{N})$ with the strong operator topology, and $\mathscr{F}(\mathscr{A})$ is as in 2.7.2. There exists a bilinear operator $\{(g, E) \to \mathbf{S}(g; dE)\}$ which maps $(BV) \times \mathscr{F}(\mathscr{A})$ into $\mathscr{A}$. Suppose that $g \in (BV)$, and let* var g *be the total variation of g. If $E \in \mathscr{F}(\mathscr{A})$, then $\mathbf{S}(g; dE) \in \mathscr{A}$ and*

$$(b_1) \qquad |\mathbf{S}(g; dE)|_r \leqslant (|g(b)| + \operatorname{var} g) \sup_{a \leqslant \lambda \leqslant b} |E(\lambda)|_r \qquad (\text{all } r \in r_0);$$

further,

(b$_2$) $\quad \mathbf{S}(g; dE) = g(b)\cdot E(b) - \oint E\cdot dg.$

Let $\chi[\alpha, b]$ *denote the characteristic function of the closed interval* $[\alpha, b]$*; if* $A \in \mathscr{A}$ *then* $A\chi[\alpha, b] \in \mathscr{F}(\mathscr{A})$ *and*

(b$_3$) $\quad \mathbf{S}(g; dA\chi[\alpha, b]) = g(\alpha)\cdot A \qquad$ (whenever $a \leqslant \alpha \leqslant b$).

If $F \in \mathscr{F}(\mathscr{A})$ *and* $F(a) = \bigcirc$ (=the zero element of $\mathscr{A}$), *then*

(b$_4$) $\quad \mathbf{S}(\chi[a, \beta]; dF) = F(\beta) \qquad$ (whenever $a \leqslant \beta \leqslant b$),

and

(b$_5$) $\quad \mathbf{S}(g; dF) = \int g(\oplus[\lambda])\cdot dF(\lambda);$

moreover, if g *is left-continuous, then*

(b$_6$) $\quad \mathbf{S}(g; dF) = \int g(\lambda)\cdot dF(\lambda).$

On the other hand, suppose that $(g_s : s \in \sigma, \succ)$ *is a generalized sequence in* (BV) *such that*

(i) $\quad \infty \neq \sup_{s\in\sigma} \operatorname{var} g_s,$

and let $h \in (BV)$ *satisfy the relation*

(ii) $\quad h(\lambda) = \lim \{g_s(\lambda) : s \in \sigma, \succ\} \qquad$ (whenever $a < \lambda \leqslant b$);

if $F(a) = \bigcirc$, *then*

(b$_7$) $\quad \mathbf{S}(h; dF) = \lim \{\mathbf{S}(g_s; dF) : s \in \sigma, \succ\}.$

Proof. See 4.3.

2.11. All the above conclusions are valid when $E \in \mathscr{E}(\mathscr{A})$ and $\mathscr{A}$ is an arbitrary *complete* locally convex Hausdorff vector space; this is borne out by the following observations. The space $\mathscr{A}$ is identifiable with the space $\mathscr{L}(C, \mathscr{A})$ (where C = the complex plane); if $\mathscr{L}(C, \mathscr{A})$ is endowed with the uniform operator topology, then $\mathscr{L}(C, \mathscr{A})$ is topologically equivalent to $\mathscr{A}$, and (v$_4$) is equivalent to (17): by identifying the topological spaces $\mathscr{A}$ and $\mathscr{L}(C, \mathscr{A})$ it follows that $\mathscr{E}(\mathscr{A}) = \mathscr{F}(\mathscr{A})$ (note that the conditions (v$_1$)–(v$_3$) imply that E is a bounded function).

3. *On Stieltjes and modified integrals*

The locally convex Hausdorff space $\mathscr{A}$ is fixed throughout. The family $\mathscr{E}(\mathscr{A})$ was defined in 2.7.

3.1 Definitions. If $a \leqslant \alpha < \beta \leqslant b$, let $[\alpha, \beta + 0)$ denote the half-open interval $[\alpha, \beta)$ when $\beta \neq b$, and set $[\alpha, b + 0) = [\alpha, b]$; thus,

$$(21) \qquad [\alpha, \beta + 0) = \begin{cases} [\alpha, \beta) \text{ if } \beta < b \\ [\alpha, b] \text{ if } \beta = b. \end{cases}$$

Let $\chi[\alpha, \beta+0)$ be the characteristic function of the interval $[\alpha, \beta+0)$. If $A \in \mathscr{A}$, then $A\chi[\alpha, \beta+0)$ is defined as in 2.8; it is the function whose value is A on the interval $[\alpha, \beta+0)$ (the value is zero elsewhere). Since $\chi[\alpha, \beta+0) \in \mathscr{F}(C)$, it is easily seen that $A\chi[\alpha, \beta+0) \in \mathscr{F}(\mathscr{A})$.

3.2. Lemma. If $A \in \mathscr{A}$, $g \in (BV)$ and $a \leqslant \alpha < \gamma \leqslant b$, then

$$\oint A\chi[\alpha, \gamma+0) \cdot dg = (g(\gamma) - g(\alpha)) \cdot A. \tag{22}$$

Proof. First, assume that $\alpha \neq a$. If $\gamma < b$ then $\chi[\alpha, \gamma+0) = \chi[a, \gamma) - \chi[a, \alpha)$, whence

$$\chi[\alpha, \gamma+0) = \chi[a, \gamma+0) - \chi[a, \alpha+0); \tag{23}$$

when $\gamma = b$ then $\chi[\alpha, b+0) = \chi[\alpha, b] = \chi[a, b] - \chi[a, \alpha)$, which shows that (23) holds for $a < \alpha \leqslant \gamma \leqslant b$; in view of (14)–(15), formula (22) now comes from (23), provided that

$$\oint \chi[a, \beta+0) \cdot dg = g(\beta) - g(a) \qquad (\text{when } a < \beta \leqslant b). \tag{24}$$

If $\alpha = a$, then (22) comes from (24). On the other hand, $\beta = b$ implies that $\chi[a, \beta+0) = \chi[a, b]$, so that (24) is immediately verified. It remains only to prove (24) in the case $a < \beta < b$.

Set $f = \chi[a, \beta+0)$, and note that $f(\lambda) = 0$ when $\lambda \geqslant \beta$. Consider the subdivision w whose range is $\{a, \beta, b\}$. If $x \in \pi(g, i)$ and $\bar{x} \geqq w$, then $\beta = \bar{x}_n$ for some $\bar{x}_n$ in the range $\{\bar{x}_k : 0 \leqslant k \leqslant m\}$ of x; but $1 \leqslant n \leqslant m-1$, and consequently the notation of (6) enables us to write

$$\Sigma f \cdot (g\Delta x) = \sum_{k=1}^{n} f(x_k{}^*) \cdot (g(\bar{x}_k) - g(\bar{x}_{k-1})) + t,$$

where $t = f(x_{n+1}^*) \cdot (g(\bar{x}_{n+1}) - g(\bar{x}_n)) + \ldots = 0$. Since $f(\lambda) = 1$ when $\lambda < \beta = \bar{x}_n$, we now see that

$$\Sigma f \cdot (g\Delta x) = \begin{cases} g(\beta) - g(a) & \text{if } x_n{}^* < \beta \\ g(\bar{x}_{n-1}) - g(a) & \text{if } x_n{}^* = \beta. \end{cases} \tag{25}$$

The proof of (24) will be concluded by establishing the existence (for $\varepsilon > 0$ arbitrary) of a subdivision u such that the relation

$$|g(\beta) - g(a) - \Sigma f \cdot (g\Delta x)| \leqslant \varepsilon \tag{26}$$

holds whenever $x \in \pi(g, i)$ and $\bar{x} \geqq u$.

First, consider the case $g \notin G$: from (11) we see that $\pi(g, i) = (\pi i)$. If $x \in (\pi i)$ and $\bar{x} \geqq w$, then $\bar{x}_{n-1} < x_n{}^* < \bar{x}_n$ (see 2.4); but $\bar{x}_n = \beta$, so that $x_n{}^* < \beta$, and (26) is now a consequence of (25).

Next, the case $g \in G$: from (11) it is seen that $\pi(g, i) = (\pi)$. From 2.5 (G) it follows the existence of a $\delta > 0$ such that

$$|g(\beta) - g(\lambda)| \leqslant \varepsilon \qquad (\text{when } 0 \leqslant \beta - \lambda < \delta). \tag{27}$$

It is easy to construct a subdivision u such that $u \geqq w$ and $||u|| < \delta$ (see

(4)); take $x \in (\pi)$ with $\bar{x} \geqq u$; it follows that $\bar{x} \geqq w$, whence the preceding considerations show that $\beta = \bar{x}_n$. From $\bar{x} \geqq u$ it results that $||x|| \leqslant ||u|| < \delta$; hence $0 \leqslant \bar{x}_n - \bar{x}_{n-1} < \delta$ and (since $\beta = \bar{x}_n$) therefore $0 \leqslant \beta - \bar{x}_{n-1} < \delta$; formula (27) now shows that

$$|g(\beta) - g(a) - (g(\bar{x}_{n-1}) - g(a))| \leqslant \varepsilon. \tag{28}$$

If $x_n^* < \beta$, then (25) implies (26). If $x_n^* = \beta$, then (25) and (28) imply (26). The proof is concluded.

3.3. Remarks. If $F \in \mathscr{E}(\mathscr{A})$ and $u = (u_\nu : 0 \leqslant \nu \leqslant \mu)$ is a subdivision, it is easily verified that the $\mathscr{A}$-valued step-function

$$F_u = \sum_{\nu=1}^{\mu} F(u_{\nu-1})\, \chi[u_{\nu-1}, u_\nu + 0) \tag{29}$$

belongs to $\mathscr{E}(\mathscr{A})$; the interval $[u_{\nu-1}, u_\nu + 0)$ was defined in (21). Note that

$$[a, b] = [a, u_1) \dots [u_{\mu-1}, b] = \bigcup_{\nu=1}^{\mu} [u_{\nu-1}, u_\nu + 0); \tag{30}$$

in effect, formula (29) defines F_u as the function whose value is $F(u_{\nu-1})$ on the interval $[u_{\nu-1}, u_\nu + 0)$. Moreover, (29) and 3.2 imply that

$$\oint F_u \cdot dg = \sum_{\nu=1}^{\mu} (g(u_\nu) - g(u_{\nu-1})) \cdot F(u_{\nu-1}). \tag{31}$$

Henceforth, we consistently presuppose that $\mathscr{A}$ is as in 2.10 and denote by $\{|\cdot|_r : r \in r_0\}$ a family of semi-norms defining the topology of $\mathscr{A}$. When $E \in \mathscr{E}(\mathscr{A})$ and $r \in r_0$ we write

$$||E||_r = \sup_{a \leqslant \lambda \leqslant b} |E(\lambda)|_r. \tag{32}$$

3.3.1. Lemma. Suppose that $F \in \mathscr{E}(\mathscr{A})$, $\varepsilon > 0$ and $r \in r_0$. There exists a subdivision $w = (w_i : 0 \leqslant i \leqslant n)$ such that

$$|F(\alpha) - F(\beta)|_r \leqslant \varepsilon \tag{33}$$

whenever α and β belong to the interval $[w_{i-1}, w_i + 0)$ and $i = 1, 2, 3, \dots, n$.

Proof. The notation $[\alpha, \beta + 0)$ was introduced in (21). Property 2.7 (v_2) implies the existence of a number $Lb \in [a, b]$ such that (33) holds when $\alpha, \beta \in [Lb, b]$; but $F(a) = F(a+0)$ (from 2.7 (v_3)), whence the existence of a number $Ra \in [a, b]$ such that (33) holds when $\alpha, \beta \in [a, Ra)$. If $Lb \leqslant Ra$, the conclusion is obtained by setting $w_1 = (Ra + Lb)/2$; the rest of this proof concerns the case $Ra < Lb$.

Suppose that $\lambda \in [Ra, Lb]$: since $a < Ra < Lb < b$, property 2.7 (v_1) ensures the existence of a number $L\lambda \in [a, b]$ such that (33) holds when $\alpha, \beta \in [L\lambda, \lambda)$, and from property 2.7 (v_3) is inferred the existence of a number $R\lambda \in [a, b]$ such that (33) holds when α and β belong to $[\lambda, R\lambda)$; in particular,

$$L\lambda < \lambda < R\lambda. \tag{34}$$

Since the family of open intervals $(L\lambda, R\lambda)$ forms an open cover of the compact set $[Ra, Lb]$, there exists a finite set $\{a_k : k \in K\} \subset [Ra, Lb]$ such that

$$[Ra, Lb] \subset \bigcup \{(La_k, Ra_k) : k \in K\}. \tag{35}$$

From (34) we have

$$La_k < a_k < Ra_k, \tag{36}$$

and the preceding remarks show that

$$\text{property (33) holds when } \alpha, \beta \in [La_k, a_k) \tag{37}$$

$$\text{property (33) holds when } \alpha, \beta \in [a_k, Ra_k). \tag{37*}$$

The conclusion is now obtained by letting w be the subdivision whose range is the following set

$$P = \{La_k : k \in K\} \cup \{a_k : k \in K\} \cup \{Ra_k : k \in K\}.$$

More precisely: set $w_0 = a$ and define w_i recursively by the formula

$$w_i = \min \{p \in P : p > w_{i-1}\}, \tag{38}$$

and set $w = (w_i : 0 \leqslant i \leqslant n)$, where

$$n = 1 + \max \{i : w_i \in [Ra, Lb]\}. \tag{39}$$

It will suffice to verify property (33) in the case $1 \leqslant i-1 \leqslant n-1$, since $[w_0, w_1 + 0) \subset [a, Ra)$, $[w_{n-1}, w_n + 0) \subset [Lb, b]$, and in view of the initial remarks. Accordingly, the rest of the proof is devoted to the case $1 \leqslant i-1 \leqslant n-1$. Since $w_{i-1} \in [Ra, Lb]$, it follows from (35) the existence of $k \in K$ with $La_k < w_{i-1} < Ra_k$. In view of (36), there are only two possibilities: either

$$La_k < w_{i-1} < a_k, \quad \text{or} \tag{40}$$

$$a_k \leqslant w_{i-1} < Ra_k. \tag{40*}$$

If (40), then $a_k > w_{i-1}$, which (since (38) and $a_k \in P$) implies that $w_i \leqslant a_k$; but then (40) shows that $[w_{i-1}, w_i)$ is contained in $[La_k, a_k)$: property (33) now comes from (37). On the other hand, if (40*), then $Ra_k > w_{i-1}$, which (since (38) and $Ra_k \in P$) implies that $w_i \leqslant Ra_k$; but then (40*) shows that $[w_{i-1}, w_i)$ is contained in $[a_k, Ra_k)$: property (33) now comes from (37*). The proof is complete.

3.4. Theorem C. *If $F \in \mathscr{E}(\mathscr{A})$ and $r \in r_0$, then*

$$0 = \lim_{u \in U} \|F - F_u\|_r. \tag{41}$$

Proof. Take $\varepsilon > 0$, and let w be as in the preceding lemma. Take any $u = (u_\nu : 0 \leqslant \nu \leqslant \mu)$ with $u \geqq w$. Our aim is to prove that $|F(\lambda) - F_u(\lambda)|_r \leqslant \varepsilon$ (for arbitrary $\lambda \in [a, b]$). From (30) it follows that λ belongs to some interval $[u_{\nu-1}, u_\nu+0)$, and (29) shows that $F_u(\lambda)=F(u_{\nu-1})$: accordingly, the proof will be concluded by showing that

$$|F(\lambda)-F(u_{\nu-1})|_r \leqslant \varepsilon. \tag{42}$$

Since $u \geqq w$, it follows that $[u_{\nu-1}, u_\nu]$ is contained in some interval $[w_{i-1}, w_i]$; since both λ and $u_{\nu-1}$ belong to $[u_{\nu-1}, u_\nu+0)$, they belong to $[w_{i-1}, w_i+0)$: thus, (42) is a consequence of 3.3.1.

3.5. Theorem. *Suppose $g \in (BV)$ and let $F \in \mathscr{E}(\mathscr{A})$ be defined as in* 2.7. *If the integral*

$$\oint F \cdot dg \tag{43}$$

exists, then it satisfies the following two relations

$$|\oint F \cdot dg|_r \leqslant (\operatorname{var} g) \cdot ||F||_r \qquad (\text{for all } r \in r_0), \tag{44}$$

$$\oint F \cdot dg = \lim_u \oint F_u \cdot dg; \tag{45}$$

see (32) *and* (29) *for the definition of $||F||_r$ and F_u. If $\mathscr{A}$ is complete, then the integral* (43) *exists.*

Proof. When $E \in \mathscr{E}(\mathscr{A})$ it is readily verified that

$$|\Sigma F \cdot (g\Delta x) - \Sigma E \cdot (g\Delta x)|_r \leqslant (\operatorname{var} g) \cdot ||F-E||_r \tag{46}$$

for all $x \in \pi(g, i)$; inequality (44) is obtained by setting $E=\bigcirc$ in (46), and from (44) it results that

$$|\oint (F-F_u) \cdot dg|_r \leqslant (\operatorname{var} g) \cdot ||F-F_u||_r,$$

whence (45) is now a consequence of Theorem C(3.4). To show that existence of (43), take $x \in \pi(g, i)$ arbitrarily, and set

$$\sigma(u, x) = \Sigma F_u \cdot (g\Delta x) \qquad (\text{all } u \in U); \tag{47}$$

consequently, by substituting $E=F_u$ in (46), we find that

$$|\Sigma F \cdot (g\Delta x) - \sigma(u, x)|_r \leqslant (\operatorname{var} g) \cdot ||F-F_u||_r, \tag{48}$$

and from Theorem C(3.4) it follows that

$$\lim_u \sigma(u, x) = \Sigma F \cdot (g\Delta x). \tag{49}$$

If $y, z \in \pi(g, i)$, then the triangle inequality shows that

$$|\Sigma F \cdot (g\Delta y) - \Sigma F \cdot (g\Delta z)|_r \leqslant A_y + A_z + |\sigma(u, y) - \sigma(u, z)|_r, \tag{50}$$

where $A_x = |\Sigma F \cdot (g\Delta x) - \sigma(u, x)|_r$ for $x=y$ and $x=z$; from (48) and (50) it now follows that

$$|\Sigma F \cdot (g\Delta y) - \Sigma F \cdot (g\Delta z)|_r \leqslant 2(\operatorname{var} g) \cdot ||F-F_u||_r + |\sigma(u, y) - \sigma(u, z)|_r. \tag{51}$$

On the other hand, from (31) it follows the existence of the integral

$$\oint F_u \cdot dg = \lim_x \sigma(u, x)$$

(the equality comes from (47) and definition 2.5(13)); consequently:

(52) $$0 = \lim_{y,z} |\sigma(u, y) - \sigma(u, z)|_r,$$

and the inequality

(53) $$\lim_{y,z} |\Sigma F \cdot (g\Delta y) - \Sigma F \cdot (g\Delta z)|_r \leqslant 2(\operatorname{var} g) \cdot ||F - F_u||_r$$

is now obtained from (52) by taking limits ($y, z \in \pi(g, i)$, $\geqq$; see 2.6) of both sides of (51). Finally, taking limits ($u \in U$, $\geqq$) of both sides of (53), Theorem C(3.4) shows that

(54) $$0 = \lim_{y,z} |\Sigma F \cdot (g\Delta y) - \Sigma F \cdot (g\Delta z)|_r.$$

As we pointed out in 2.6, the completeness hypothesis on $\mathscr{A}$ implies that (54) ensures the existence of

$$\lim_x \Sigma F \cdot (g\Delta x) = \oint F \cdot dg;$$

the equality is from definition 2.5(13).

3.6. Theorem. *Let $\mathfrak{M}$ and $\mathscr{N}$ be locally convex Hausdorff vector spaces, and suppose that $\mathscr{N}$ is complete; if $\mathscr{A} = \mathscr{L}(\mathfrak{M}, \mathscr{N})$ and $F \in \mathscr{F}(\mathscr{A})$, then the integral* (43) *exists.*

Proof. The family $\mathscr{F}(\mathscr{A})$ was defined in 2.7.2; as in 2.7.1, let $\{||\cdot||_j : j \in j_0\}$ be a family of semi-norms determining the topology of $\mathscr{N}$. Take $E \in \mathscr{F}(\mathscr{A})$ and let m be an arbitrary element of $\mathfrak{M}$; obviously,

(55) $$||(E(\lambda) - E(\beta))(m)||_j = ||E(\lambda)(m) - E(\beta)(m)||_j.$$

In view of (55) and (16), the existence of the limit $E(\beta - 0)$ is a consequence of 2.7 (v_1), whence it follows that the $\mathscr{N}$-valued function $\{\lambda \to E(\lambda)(m)\}$ (denoted $E(\cdot)m$) also satisfies 2.7 (v_1). This type of remark (twice repeated) shows that $E(\cdot)m \in \mathscr{E}(\mathscr{N})$: replacing r_0, $\mathscr{A}$, F in 3.5 respectively by j_0, $\mathscr{N}$, $E(\cdot)m$, we may now infer from 3.5 that the integral

$$\oint E(\cdot)m \cdot dg = \lim_x \Sigma E(\cdot)m \cdot (g\Delta x)$$

exists and defines an element $A(m)$ of $\mathscr{N}$. Therefore, from definition 2.5(13) we see that

(56) $$0 = \lim_x ||A(m) - \Sigma E(\cdot)m \cdot (g\Delta x)||_j \qquad (\text{any } j \in j_0);$$

and

(57) $$A(m) = \oint E(\cdot)m \cdot dg \qquad (\text{all } m \in \mathfrak{M}).$$

Again replacing $r_0, \mathscr{A}, F$ in 3.5 respectively by $j_0, \mathscr{N}, E(\cdot)m$, it follows from (57), (44) and (32) that

$$(58) \qquad ||A(m)||_j = ||\oint E(\cdot)m\cdot dg||_j \leqslant (\operatorname{var} g)\cdot \sup_{a\leqslant\lambda\leqslant b} ||E(\lambda)(m)||_j.$$

We may infer from 2.7.2 (v_4) and (58) the existence of a finite family $M_j \subset M_{00}$ (here M_{00} is a family of semi-norms determining the topology of $\mathfrak{M}$), such that

$$(59) \quad ||A(m)||_j \leqslant (\operatorname{var} g)\cdot j[E]\cdot \max\{p(m) : p \in M_j\} \quad (\text{all } j \in j_0,\ m \in \mathfrak{M}).$$

The operator $A = \{m \to A(m)\}$ is clearly a linear mapping of $\mathfrak{M}$ into $\mathscr{N}$, and (59) obviously implies that $A \in \mathscr{L}(\mathfrak{M}, \mathscr{N})$ (see 2.7.1): thus, $A \in \mathscr{A}$.

On the other hand, it is immediately verified that

$$||A(m) - \Sigma E(\cdot)m\cdot(g\Delta x)||_j = ||(A - \Sigma E\cdot(g\Delta x))(m)||_j,$$

whence it now follows from (56) and (16) that

$$A = \lim_x \Sigma E\cdot(g\Delta x) = \oint E\cdot dg;$$

the limit ($x \in \pi(g, i), \geqq$) is in the topology of $\mathscr{A}$, and the second equality comes from (13). Since $A \in \mathscr{A}$, the proof is now completed.

3.7. Theorem. *If $g \in (BV)$ and $E \in \mathscr{F}(\mathscr{A})$, the equation*

$$(60) \qquad \mathbf{S}(g; dE) = g(b)\cdot E(b) - \oint E\cdot dg$$

defines an element of $\mathscr{A}$ such that

$$(1) \qquad |\mathbf{S}(g; dE)|_r \leqslant (|g(b)| + \operatorname{var} g)\cdot ||E||_r \qquad (\text{all } r \in r_0).$$

When $a \leqslant \alpha \leqslant b$, let $\chi[\alpha, b]$ denote the characteristic function of the closed interval $[\alpha, b]$; if $A \in \mathscr{A}$ then $A\chi[\alpha, b] \in \mathscr{F}(\mathscr{A})$ and

$$(2) \qquad \mathbf{S}(g; dA\chi[\alpha, b]) = g(\alpha)\cdot A.$$

On the other hand, suppose that $(g_s : s \in \sigma, \succ)$ is a generalized sequence in (BV) such that

$$(\text{i}) \qquad \infty \neq \sup_s \operatorname{var} g_s,$$

and let $h \in (BV)$ satisfy the relation

$$(\text{ii}) \qquad h(\lambda) = \lim \{g_s(\lambda) : s \in \sigma, \succ\} \quad (\text{whenever } a < \lambda \leqslant b);$$

if $F \in \mathscr{F}(\mathscr{A})$ and $F(a) = \bigcirc$, then relation 2.10 (b_7) helds, and

$$(b_8) \qquad \oint F\cdot dh = \lim_s \oint F\cdot dg_s.$$

Proof. The existence of the integral in (60) follows from 3.6, and (1) is an immediate consequence of (44). We begin by proving (2). From

(60) and (18) it follows that

$$\mathbf{S}(g;\, dA\chi[\alpha, b]) = (g(b)\chi[\alpha, b](b))\cdot A - \oint A\chi[\alpha, b]\cdot dg. \tag{3}$$

Note that $\chi[\alpha, b](b)=1$; setting $\gamma=b$ in 3.2, conclusion (2) now comes directly from (3).

Next, to prove (b_7)–(b_8). Since (ii) implies the convergence of $g_s(b)$ to $h(b)$, conclusion (b_7) is an immediate consequence of (b_8) and (60); it only remains to prove (b_8).

Set $h_s = h - g_s$ and note that

$$0 = \lim_s h_s(\lambda) \qquad (\text{whenever } \lambda > a). \tag{4}$$

Take an arbitrary $u \in U$. From the linearity properties (19)–(20) it is readily verified that

$$\oint F\cdot dh_s = \oint (F - F_u)\cdot dh - \oint (F - F_u)\cdot dg_s + \oint F_u\cdot dh_s. \tag{5}$$

From (i) it follows the existence of a constant c_0 such that $c_0 \geqslant \operatorname{var} h + \operatorname{var} g_s$ whnever $s \in \sigma$. Thus, from (44) and (5) we have

$$|\oint F\cdot dh_s|_r \leqslant c_0 ||F - F_u||_r + |\oint F_u\cdot dh_s|_r. \tag{6}$$

Since $F(u_{\nu-1}) = F(a) = \bigcirc$ for $\nu = 1$, the corresponding term vanishes in (29), and from (31) it follows that

$$|\oint F_u\cdot dh_s|_r \leqslant ||F||_r\, (|(h_s\Delta u_2)| + \ldots + |(h_s\Delta u_\mu|), \tag{7}$$

where $(h_s\Delta u_\nu) = h_s(u_\nu) - h_s(u_{\nu-1})$; since $a < u_1 < u_\nu$, it follows from (4) that

$$0 = \lim_s |(h_s\Delta u_\nu)| \qquad (\nu = 2, 3, \ldots, \mu). \tag{8}$$

Taking limits ($s \in \sigma$, $\succ$) of both sides of (6), we obtain from (7)–(8) that

$$\lim_s |\oint F\cdot dh_s|_r \leqslant c_0 ||F - F_u||_r, \tag{9}$$

and the conclusion (b_8) now comes from Theorem C(3.4) by taking limits ($u \in U$, $\geqq$) of both sides of (9).

3.8. Remarks. In case the range-space $\mathscr{A}$ is complete, the existence of the integral (43) — which is the crucial point of this paper — could have been obtained by extending theorem 15.1 in [8, p. 69]; also, rather than proving it directly, conclusion (b_8) could have been inferred from some of the most general Helly theorems in the literature (e.g., [17] and [8, p. 71]). Let $\mathscr{T}$ be the topology defined by the family $\{||\cdot||_r : r \in r_0\}$ of semi-norms defined by relation (32): Theorem C(3.4) states that $\mathscr{E}(\mathscr{A})$ is the $\mathscr{T}$-closure of the linear span of all functions of the form $A\chi[\alpha, \beta+0)$ (where $A \in \mathscr{A}$ and $a \leqslant \alpha < \beta \leqslant b$).

4. *On Stieltjes and Cauchy–Stieltjes integrals*

This section consists of two lemmas followed by the proofs of Theorem B(2.10) and Theorem A. It will be convenient to denote by $e(\beta)$ the characteristic function $\chi[a, \beta]$ of the closed interval $[a, \beta]$. Note that

(10) $$e(\beta)-e(\alpha)=\chi(\alpha, \beta] \text{ if } a\leqslant\alpha<\beta\leqslant b.$$

4.1. Lemma. If $F \in \mathscr{F}(\mathscr{A})$ and $F(a)=\bigcirc$, then

(b_4) $$\mathbf{S}(e(\beta); dF)=F(\beta) \qquad (\text{when } a\leqslant\beta\leqslant b).$$

Proof. Definition (60) shows that

(11) $$\mathbf{S}(e(\beta); dF)=e(\beta)(b)\cdot F(b)-\oint F\cdot de(\beta).$$

If $\beta=b$, then $e(\beta)$ is the constant 1, and (b_4) somes from (11). If $\beta<b$ then $e(\beta)(b)=0$; it needs to be shown that

(12) $$\oint F\cdot de(\beta)=-F(\beta),$$

and we have only to prove (12) in the case $a\leqslant\beta<b$. Set $g=e(\beta)$ and let F_u be as in (29); from (31) we have

(13) $$\oint F_u\cdot dg=\sum_{\nu=1}^{\mu}(g(u_\nu)-g(u_{\nu-1}))\cdot F(u_{\nu-1}).$$

Clearly, β belongs to a unique interval $[u_{n-1}, u_n)$ outside of which g is constant and $g(u_n)-g(u_{n-1})=0-1$; from (13) it therefore follows that

(14) $$\oint F_u\cdot dg=-F(u_{n-1})=-F_u(\beta);$$

the last equality comes from 3.3 and from the fact that $\beta\in[u_{n-1}, u_n)$. It may now be inferred from (45) and (14) that

(15) $$\oint F\cdot dg=-\lim_u F_u(\beta)=-F(\beta);$$

the last equality comes from (41). Since (15) implies (12), the proof of (b_4) is complete.

4.2. Lemma. *Suppose that* $g\in(BV)$. *If* $x=(\bar{x}, x^*)$ *is a marked partition with range* $\{\bar{x}_k : 0\leqslant k\leqslant m\}$, *then the function*

(16) $$g_x=\sum_{k=1}^{m} g(x_k{}^*)\cdot(e(\bar{x}_k)-e(\bar{x}_{k-1}))$$

belongs to (BV). *Moreover, if* σ *is the family* $(\pi\oplus)$ *that was defined in* 2.4, *then*

(17) $$g(\beta)=\lim\{g_x(\beta) : x\in\sigma, \geqq\} \qquad (\text{when } a<\beta\leqslant b);$$

in case $g\in G$, *then* (17) *holds when* $\sigma=(\pi)$.

Proof. The family (π) consists of all marked partitions (see 2.2), and G is the family of all functions $g\in(BV)$ such that

(18) $$g(\beta-0)=g(\beta) \qquad (\text{when } a<\beta\leqslant b).$$

Let w be the subdivision whose range is $\{a, \beta, b\}$, and suppose that $x \in (\pi)$, $\bar{x} \geqq u$, $u \in U$ and $u \geqq w$. It follows from (16) and (10) that

$$g_x = \sum_{k=1}^{m} g(x_k^*) \cdot \chi(\bar{x}_{k-1}, \bar{x}_k]; \tag{19}$$

here (as well as in (16)), the terms are of the form $g(x_k^*) \cdot h = \{\lambda \to g(x_k^*)h(\lambda)\}$. Thus, g_x is the function whose value is $g(x_k^*)$ on the interval $(\bar{x}_{k-1}, \bar{x}_k]$. From $\bar{x} \geqq u$ and $u \geqq w$ it follows that $\bar{x} \geqq w$, whence

$$\beta = \bar{x}_n \qquad \text{(for some } \bar{x}_n \text{ in the range of } x); \tag{20}$$

but (19) then shows that $g_x(\beta) = g(x_n^*)$. Accordingly, (17) shall be established by showing that an arbitrary $\varepsilon > 0$ gives rise to a subdivision u such that

$$|g(x_n^*) - g(\beta)| \leqslant \varepsilon \qquad (\text{whenever } \bar{x} \geqq u). \tag{21}$$

In the case $\sigma = (\pi\oplus)$, take $x \in (\pi\oplus)$ with $\bar{x} \geqq u = w$; from definition 2.4 we see that $x_n^* = \bar{x}_n$, and from (20) we obtain $x_n^* = \beta$, whence $g(x_n^*) - g(\beta) = 0$: relation (21) holds.

Finally, the case $\sigma = (\pi)$ and $g \in G$. From (18) is inferred the existence of a $\delta > 0$ such that

$$|g(\lambda) - g(\beta)| \leqslant \varepsilon \qquad (\text{when } 0 \leqslant \beta - \lambda < \delta); \tag{22}$$

let u be a subdivision such that $u \geqq w$ and $||u|| < \delta$. Take $x \in (\pi)$ with $\bar{x} \geqq u$; from $\bar{x} \geqq u$ it follows that $||\bar{x}|| < \delta$, so that $|\bar{x}_{n-1} - \bar{x}_n| < \delta$; but $\bar{x}_{n-1} \leqslant x_n^* \leqslant \bar{x}_n = \beta$ (from (20)), whence $0 \leqslant \beta - x_n^* < \delta$; conclusion (21) now comes from (22).

4.3. Proof of Theorem B(2.10). The bilinearity property comes immediately from the definition (60) of $\boldsymbol{S}(g; dE)$. In view of 3.7, and since (b_4) was proved in 4.1, it only remains to establish (b_5)–(b_6). To that effect, let F be as in 4.1; it must be shown that, if $g \in (BV)$ then

$$\boldsymbol{S}(g; dF) = \int g(\oplus[\lambda]) \cdot dF(\lambda), \tag{b_5}$$

and, if $g \in G$ then

$$\boldsymbol{S}(g; dF) = \int g(\lambda) \cdot dF(\lambda). \tag{b_6}$$

It will suffice to deal with the case where g belongs to the famliy V of all monotone-increasing real-valued functions on $[a, b]$; the validity of (b_5)–(b_6) can then be inferred as follows. Write $T_1(g) = \boldsymbol{S}(g; dF)$ and let $T_2(g)$ and $T_3(g)$ denote the respective right-hand sides of (b_5) and (b_6); clearly

$$T_p\left(\sum_i t_i \cdot g^i\right) = \sum_i t_i \cdot T_p(g^i) \tag{23}$$

when $p = 1, 2, 3$, $g^i \in (BV)$, and for finite sequences $(t_i : i)$ of complex numbers. Since any $g \in (BV)$ can be expressed in the form $g = t_1 g^1 + \ldots + t_4 g^4$

with $g^i \in V$, the desired conclusion $T_1(g) = T_p(g)$ will follow from (23) if we prove that $T_1(g) = T_p(g)$ (i.e., formulas (b_5)–(b_6)) for an arbitrary $g \in V$. It is easily verified that

$$\text{if } h \in V, \text{ then } \operatorname{var} h \leqslant 2\|h\|, \tag{24}$$

(where $\|h\| = \sup\{|h(\lambda)|;\, \lambda \in [a, b]\}$). Take $x \in (\pi)$ and $\lambda \in [a, b]$; from (19) it follows that $g_x(\lambda) = g(x_k{}^*)$, and therefore $\|g_x\| \leqslant \|g\|$. It is not hard to check that $g_x \in V$. Consequently, (24) implies that $\operatorname{var} g_x \leqslant 2\|g\|$: in view of 4.2(17) and 3.7, we may now infer from (b_7) that

$$\mathbf{S}(g; dF) = \lim\{\mathbf{S}(g_x; dF) : x \in \sigma, \geqq\}; \tag{25}$$

here $\sigma = (\pi\oplus)$: in case $g \in G$, then 4.2(17) likewise implies the validity of (25) when $\sigma = (\pi)$. From 4.2(16), (23) and 4.1 it follows that

$$\mathbf{S}(g_x; dF) = \sum_{k=1}^{m} g(x_k{}^*) \cdot (F(\bar{x}_k) - F(\bar{x}_{k-1})) = \Sigma g \cdot (F\Delta x) \tag{26}$$

(the last equality brings in the notation that was introduced in 2.3); a combination of (25) and (26) shows that

$$\mathbf{S}(g; dF) = \lim\{\Sigma g \cdot (F\Delta x) : x \in \sigma, \geqq\}. \tag{27}$$

If $g \in (BV)$, set $\sigma = (\pi\oplus)$; in view of 2.4(8), conclusion (b_5) now comes from (27). If $g \in G$, set $\sigma = (\pi)$; in view of 2.4(7), conclusion (b_6) also comes from (27).

4.4. Proof of Theorem A. The conclusions (iii)–(iv) of Theorem A (in § 1) are now at hand: indeed, 2.10 (b_7) and 2.10 (b_5) imply (iii), while 3.7 (b_8) gives (iv) (in view of definition 2.5).

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